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FORMULATION AND SOLUTION OF DYNAMIC PROBLEMS OF ELASTIC ROD SYSTEMS SUBJECTED TO BOUNDARY CONDITIONS DESCRIBED BY MULTIVALUED RELATIONS

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The dynamic behavior of rod systems under the action of external force factors described by multivalued (subdifferential) relations is studied. The mathematical formulation of the problem is given in the form of a dynamic quasivariational inequality. With the use of the Newmark difference scheme, successive approximations, and finite-element discretization, the problem is reduced to minimization of a convex nonsmooth finite-dimensional functional with respect to velocities at each time step. Introduction of auxiliary variables by the method of a modified Lagrangian reduces the problem of minimization of this functional to a sequence of smooth problems of nonlinear programming. The algorithm is verified using the numerical solution for a problem with one degree of freedom. The algorithm proposed is used to calculate the rods of deep-well pumps.

Key words: *dynamic problems, rod systems, finite-element discretization.*

Introduction. The current status of computers and numerical methods makes it possible to solve problems that require much computational effort. This class includes problems of dynamic systems subject to boundary conditions described by multivalued relations. Their effective solution requires special approaches, for example, those based on variational inequalities.

The dynamic behavior of mechanical systems with discontinuous nonlinearities has been studied in many papers where the main attention was focused on oscillations of systems with Coulomb friction. Most of the studies deal with systems with one or several degrees of freedom [1–4]. In [5–7], periodic solutions are obtained and their stability is studied for systems with discontinuous nonlinearities and an arbitrary number of degrees of freedom. In these studies, the phase space of the system is divided into subdomains in which the equations governing the behavior of the system are continuous. According to this approach, one should construct boundaries between the subdomains and determine the points at which the phase trajectory intersects these boundaries, which is a difficult problem for systems with a large number of degrees of freedom.

The problem of oscillation of a system in the presence of friction can be solved as a dynamic boundary-value problem by using the solution for a linear system similar to the system considered as initial conditions and tracing this solution for several periods.

One of the universal and mathematically well-posed formulations is the formulation of the boundary-value problem with nonlinearities of the friction type in the form of variational and quasivariational inequalities. The theory of these inequalities is described in detail in [8–12], and their applications in mechanics are considered in [13–15].

In the present paper, we study the motion of an elastic rod system taking into account external force factors described by multivalued relations.

1. Formulation of the Problem. We consider a system in the form of a heavy elastic rod located in a curvilinear channel with a viscous fluid flow. The distributions of pressure, density, and flow rate of the fluid along the channel are assumed to be known. The channel diameter is several orders smaller than the radius of curvature

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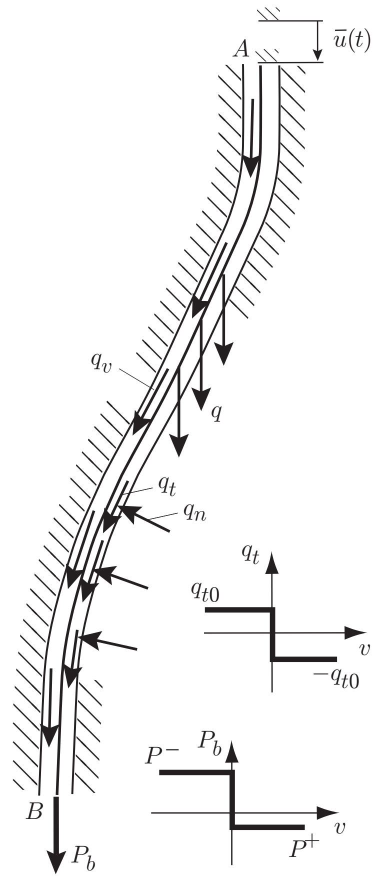

Fig. 1. Geometry of the problem.

of the channel axis and commensurable with the rod diameter. The rod is confined by the channel walls and can move only along the channel axis. Figure 1 shows the geometry of the problem. The upper end of the rod (point A in Fig. 1) performs periodic motion according to a specified law $\bar{u}(t)$, and the lower end (point B in Fig. 1) is subjected to a force which depends on the direction of motion of the rod:

$$P_B(\dot{u}_B) = \begin{cases} P^-, & \dot{u}_B < 0, \\ [P^+, P^-], & \dot{u}_B = 0, \\ P^+, & \dot{u}_B > 0, \end{cases} \qquad P^- > P^+. \tag{1}$$

Here $\dot{u}_B$ is the velocity of the lower end of the rod and P^- and P^+ are the forces that act on the lower end of the rod as it moves up and down, respectively.

When moving in the fluid, the rod is subjected to the action of the viscous force whose intensity per unit length (with allowance for the sign) q_v is calculated by the formula

$$q_v = q_Q - C_v \dot{u},$$

where q_Q is the component associated with the flow rate of the fluid in each section of the channel and C_v is the coefficient of hydrodynamic resistance to rod motion.

The effect of hydrostatic pressure acting on the rod from the fluid is taken into account in the following manner. In the case where uniform hydrostatic pressure acts on a portion of the rod, the resultant vector and resultant moment of the force system vanish, whereas the displacements due to the axial strain produced by uniform hydrostatic pressure are small. Therefore, the strain and pressing force that acts on the wall from the rod portion whose lateral surface is subjected to the pressure p_f in the presence of an axial force N are assumed to be equivalent to the strain and pressing force for the pressure-free rod. For the equivalent axial force, we obtain

$$N_{\text{eq}} = N + p_f A \tag{2}$$

(A is the cross-sectional area of the rod portion considered).

As a result of interaction with the channel walls, the sliding-friction force acts along the rod in the direction opposite to rod motion. The magnitude of the sliding-friction force per unit length is

$$q_{t0} = f|q_n|.$$

Here f is the friction coefficient and q_n is the pressing force of the rod to the channel walls per unit length. The vector of this force is normal to the rod axis and has no axial component. The magnitude of this force depends on the current stress-strain state of the rod:

$$q_n = \sqrt{\left((q - q_0)\sin\alpha + N_{\mathrm{eq}}\,\frac{\partial \alpha}{\partial x}\right)^2 + \left(N_{\mathrm{eq}}\,\frac{\partial \theta}{\partial x}\,\sin\alpha\right)^2}\,.$$

Here q is the weight of the rod portion of unit length, α is the angle between the tangent to the rod axis and the vertical, θ is the azimuth (angle determining the direction of the projection of the tangent to the rod axis onto the horizontal plane), and q_0 is the weight of the fluid expelled by the rod portion of unit length.

It should be noted that the main difficulties arising in studying this system are due to the presence of the Coulomb-friction force and the force acting on the lower end of the rod. These forces are related to velocities by multivalued relations: zero velocity corresponds to an interval of force values. These relations can be written in a subdifferential form. Let $f(x)$ be a convex function of n variables. By definition, the quantity y belongs to the set $\partial f(x)$ called the subdifferential of the function $f(x)$ if the following inequality holds:

$$f(x_1) - f(x) \geqslant (y, x_1 - x)_{\mathbb{R}^n} \qquad \forall y \in \mathbb{R}^n.$$

In the one-dimensional case considered, the scalar product can be replaced by the ordinary product $(y, x_1 - x)_{\mathbb{R}} = y(x_1 - x)$.

Panagiotopoulos [10] showed that the subdifferential of a function in the one-dimensional case can be written as

$$\partial f(x) = \{y \in \mathbb{R} \mid f'_-(x) \leqslant y \leqslant f'_+(x)\},$$

where $f'_-(x)$ and $f'_+(x)$ are the left- and right-sided derivatives at the point x, respectively. In this case, for the force acting on the lower end of the rod, relation (1) can be written with allowance for (2) as

$$-P_{B,\mathrm{eq}} \in \partial j_B(\dot{u}_B). \tag{3}$$

Here $j_B(\dot{u}_B)$ is a convex nonsmooth function, which is called the superpotential and is determined by the formula

$$j_B(\dot{u}_B) = \sup_{u^* \in K_B} (u^* \dot{u}_B),$$

$$K_B = [-P^-_{\mathrm{eq}}, -P^+_{\mathrm{eq}}], \qquad P^-_{\mathrm{eq}} = P^- + p_B A_B, \quad P^+_{\mathrm{eq}} = P^+ + p_B A_B.$$

The subdifferential relations for the distributed force q_t arising due to Coulomb friction are written in the form

$$-q_t \in \partial j_{t,q_n(u)}(\dot{u}), \tag{4}$$

where

$$j_{t,q_n(u)}(\dot{u}) = \mu|q_n(u)||\dot{u}| = \sup_{u^* \in K_t(u)} (u^* \dot{u}),$$

$$K_t(u) = [-\mu|q_n(u)|, \mu|q_n(u)|] = [-q_{t0}(u), q_{t0}(u)].$$

According to the definition of the subdifferential, relations (3) and (4) can be written as the variational inequality

$$j_{B,\mathrm{eq}}(v) - j_{B,\mathrm{eq}}(\dot{u}_B) \geqslant -P_{B,\mathrm{eq}}(v - \dot{u}_B) \qquad \forall v \in \mathbb{R} \tag{5}$$

and the quasivariational inequality

$$j_{t,q_n(u)}(v) - j_{t,q_n(u)}(\dot{u}) \geqslant -q_t(v - \dot{u}) \qquad \forall v \in \mathbb{R}. \tag{6}$$

To construct the solution of the problem, we use the virtual work principle, which takes the form

$$\int_L \rho \ddot{u} w A \, dx + \int_L C_v \dot{u} w \, dx + \int_L N_{\text{eq}}(u)\varepsilon(w) \, dx$$

$$= \int_L q_t(\dot{u}) w \, dx + \int_L q_Q w \, dx + \int_L \Big(\rho g \cos\alpha - \frac{\partial p_f}{\partial x}\Big) w A \, dx + P_{B,\text{eq}}(\dot{u}_B) w_B - p_A A_A w_A, \tag{7}$$

where $P_{B,\text{eq}} = P_B + p_B A_B$, ρ is the density of the rod material, A_A and A_B are the cross-sectional areas at the points A and B, respectively, p_A and p_B are the fluid pressures at the points A and B, respectively, and w is the trial function equal to zero at the point where the displacement is specified. Using the virtual work principle (7), taking into account the subdifferential boundary conditions (5) and (6), and writing the trial function w in the form $w = v - \dot{u}$, we obtain the following formulation of the problem: it is required to find the displacement field $u(t)$ that satisfies the dynamic quasivariational inequality

$$\int_L \rho \ddot{u}(v - \dot{u}) A \, dx + \int_L C_v \dot{u}(v - \dot{u}) \, dx + \int_L N_{\text{eq}}(u)\varepsilon(v - \dot{u}) \, dx + j_B(v_B) - j_B(\dot{u}_B) + \Phi_{t,q_n(u)}(v) - \Phi_{t,q_n(u)}(\dot{u})$$

$$- \int_L q_Q (v - \dot{u}) \, dx - \int_L \Big(\rho g \cos\alpha - \frac{\partial p_f}{\partial x}\Big)(v - \dot{u}) A \, dx + p_A A_A (v_A - \dot{u}_A) \geqslant 0 \qquad \forall v \in U, \tag{8}$$

initial conditions for u and $\dot{u}$, and $\dot{u}(t) \in U$. Here U is the set of admissible velocities and

$$\Phi_{t,q_n(u)}(v) = \int_L j_{t,q_n(u)}(v) \, dx.$$

We write the quasivariational inequality (8) in the form

$$(\rho A \ddot{u}, v - \dot{u}) + (C_v \dot{u}, v - \dot{u}) + a(u, v - \dot{u}) + j_B(v_B) - j_B(\dot{u}_B) + \Phi_{t,q_n(u)}(v) - \Phi_{t,q_n(u)}(\dot{u}) \geqslant l(v - \dot{u}) \qquad \forall v \in U, \tag{9}$$

where $(u, v) = \int_L uv \, dx$ is the scalar product, $a(u, v)$ is the bilinear form, and $l(v)$ is the linear functional.

2. Numerical Solution of the Quasivariational Inequality. The quasivariational inequality (9) is numerically as follows. For time discretization, we use the Newmark scheme

$$u^{(n+1)} = u^{(n)} + \Delta t \dot{u}^{(n)} + (1/2 - \beta)(\Delta t)^2 \ddot{u}^{(n)} + \beta(\Delta t)^2 \ddot{u}^{(n+1)},$$

$$\dot{u}^{(n+1)} = \dot{u}^{(n)} + (1 - \gamma)\Delta t \ddot{u}^{(n)} + \gamma \Delta t \ddot{u}^{(n+1)},$$

where $u^{(i)}$, $\dot{u}^{(i)}$, and $\ddot{u}^{(i)}$ are the displacements, velocities, and accelerations at the time t_i, respectively, and $\Delta t = t_{n+1} - t_n$ is the time step.

Expressing the displacements and accelerations at the time t_{n+1} in terms of $\dot{u}^{(n+1)}$ and quantities $u^{(n)}$, $\dot{u}^{(n)}$, and $\ddot{u}^{(n)}$ calculated at the previous step, substituting them into (9), and using the properties of the bilinear form $a(\,\cdot\,,\cdot\,)$, scalar product, and linear functional $l(\,\cdot\,)$, we obtain the following formulation of the problem: it is required to find $\dot{u}^{(n+1)} \in U_{n+1}$ that satisfies the quasivariational inequality

$$\hat{a}(\dot{u}^{(n+1)}, v - \dot{u}^{(n+1)}) + j_B(v_B) - j_B(\dot{u}_B^{(n+1)}) + \Phi_{t,q_n(u^{(n+1)})}(v) - \Phi_{t,q_n(u^{(n+1)})}(\dot{u}^{(n+1)}) \geqslant \hat{l}(v - \dot{u}^{(n+1)}) \quad \forall v \in U_{n+1}.$$

Here

$$\hat{a}(u, v) = \frac{\rho}{\gamma \Delta t}(Au, v) + (C_v u, v) + \frac{\beta \Delta t}{\gamma} a(u, v), \qquad \hat{l}(v) = l(v) + \rho(\tilde{w}, v) - a(\tilde{u}, v),$$

and $\tilde{u}$ and $\tilde{w}$ are auxiliary quantities depending on the displacements, velocities, and accelerations calculated at the previous step. This inequality involves nonsmooth functionals depending on the pressing force, which, in turn, depends on the desired solution. This difficulty can be overcome using the approach proposed in [12–14]. In these studies, it is shown that the solution of the quasivariational inequality can be constructed by solving a succession of variational inequalities of the form

$$\hat{a}(\dot{u}^{[k+1]}, v - \dot{u}^{[k+1]}) + j_B(v_B) - j_B(\dot{u}_B^{[k+1]}) + \Phi_{t,q_n(u^{[k]})}(v) - \Phi_{t,q_n(u^{[k]})}(\dot{u}^{[k+1]}) \geqslant \hat{l}(v - \dot{u}^{[k+1]}) \qquad \forall v \in U_{n+1},$$

in which the pressing force and, hence, the friction force are calculated on the basis of the solution found at the previous iteration. Here $\dot{u}^{[k]}$ is the kth approximation of the velocity field $\dot{u}^{(n+1)}$ and $u^{[k]}$ is the kth approximation of the displacement field. As the initial approximation, we use the velocity field determined at the previous step: $\dot{u}^{[0]} = \dot{u}^{(n)}$.

For a reasonably small time step, one iteration can be sufficient to obtain an accurate solution. In this case, the quasivariational inequality can be replaced by the variational inequality

$$\hat{a}(\dot{u}^{(n+1)}, v - \dot{u}^{(n+1)}) + j_B(v_B) - j_B(\dot{u}_B^{(n+1)}) + \Phi_{t,q_n(u^{(n)})}(v) - \Phi_{t,q_n(u^{(n)})}(\dot{u}^{(n+1)}) \geqslant \hat{l}(v - \dot{u}^{(n+1)}) \quad \forall v \in U_{n+1}, \tag{10}$$

where the pressing force is calculated from the solution obtained at the previous time step.

The solution of the variational inequality (10) is equivalent to minimization of the nonsmooth functional

$$J(\dot{u}^{(n+1)}) = (1/2)\hat{a}(\dot{u}^{(n+1)}, \dot{u}^{(n+1)}) + j_B(\dot{u}_B^{(n+1)}) + \Phi_t(\dot{u}^{(n+1)}) - \hat{l}(\dot{u}^{(n+1)}) \quad \text{for} \quad \dot{u}^{(n+1)} \in U_{n+1}.$$

Using the Newmark scheme, we write the set of admissible velocities U in the form

$$U_{n+1} = \{\dot{u}\}$$

with $\dot{u} = \hat{u}$ at the point A. Here $\hat{u} = \gamma(\bar{u}(t_{n+1}) - \tilde{u})/(\beta \Delta t)$ and $\bar{u}(t)$ is the specified displacement of the upper point A of the rod at the time t.

To implement the algorithm, the problem should be reduced to a finite-dimensional problem. For this purpose, the rod is divided into finite elements. In this case, the functionals in the variational inequalities can be written in the matrix form

$$a(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{u}^{\mathrm{t}} K \boldsymbol{v}, \qquad \hat{a}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{u}^{\mathrm{t}} K_* \boldsymbol{v}, \qquad (A\rho \ddot{\boldsymbol{u}}, \boldsymbol{v}) = \ddot{\boldsymbol{u}}^{\mathrm{t}} M \boldsymbol{v},$$

$$(C_v \dot{\boldsymbol{u}}, \boldsymbol{v}) = \dot{\boldsymbol{u}}^{\mathrm{t}} C \boldsymbol{v}, \qquad l(\boldsymbol{v}) = \boldsymbol{b}^{\mathrm{t}} \boldsymbol{v}, \qquad \hat{l}(\boldsymbol{v}) = \boldsymbol{b}_*^{\mathrm{t}} \boldsymbol{v},$$

where

$$K_* = \frac{\beta \Delta t}{\gamma} K + \frac{1}{\Delta t \gamma} M + C, \qquad \boldsymbol{b}_* = \boldsymbol{b} + M\tilde{\boldsymbol{w}} - K\tilde{\boldsymbol{u}}.$$

Here K, M, and C are the stiffness, mass, and damping matrices, respectively, $\boldsymbol{b}$ is the vector of external forces, and $\tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{w}}$ are vectors whose components are the values of $\tilde{u}$ and $\tilde{w}$ at discrete points.

As a result, the problem reduces to minimization of the finite-dimensional functional

$$J^h(\boldsymbol{v}) = \hat{a}^h(\boldsymbol{v}, \boldsymbol{v})/2 + \Phi(\boldsymbol{v}) - \hat{l}^h(\boldsymbol{v}) \quad \text{for} \quad \boldsymbol{v} \in U.$$

According to [9], the discrete form of the nonsmooth functional $\Phi(\boldsymbol{v})$ is calculated by the formula

$$\Phi(\boldsymbol{v}) = \sum_{i=0}^{n} \Phi_i^h(v_i),$$

where

$$\Phi_i^h(v_i) = \sup_{u^* \in \bar{K}_i} (u^* v_i), \qquad \bar{K}_i = \begin{cases} [-Q_i, Q_i], & i = 0, \ldots, n-1, \\ [Q^-, Q^+], & i = n, \end{cases}$$

$$Q_i = \begin{cases} (f/2)|q_{n,0}|\Delta l_0, & i = 0, \\ (f/2)(|q_{n,i-1}|\Delta l_{i-1} + |q_{n,i}|\Delta l_i), & i = 1, \ldots, n-1, \end{cases}$$

$$Q^- = -P_{\mathrm{eq}}^- - (f/2)|q_{n,n-1}|\Delta l_{n-1}, \qquad Q^+ = -P_{\mathrm{eq}}^+ + (f/2)|q_{n,n-1}|\Delta l_{n-1},$$

Δl_i is the length of the ith element, $q_{n,i}$ is the pressing force acting on the ith element, v_i is the ith component of the vector $\boldsymbol{v}$, and n is the total number of elements.

For discretization, the set of admissible velocities has the form

$$U = \{\boldsymbol{v} \in \mathbb{R}^{n+1} \mid v_0 = \hat{u}\}. \tag{11}$$

Condition (11) is a simple constraint.

Bertsekas [16] showed that the problem of minimization of a nonsmooth functional can be reduced to a sequence of smooth problems of nonlinear programming by the modified Lagrangian method (multiplier method).

For this purpose, additional variables z_i are introduced and the problem is formulated as follows: find the values of $\boldsymbol{v}$ and $\boldsymbol{z}$ that minimize the functional

$$J_0(\boldsymbol{v}) + \sum_{i=0}^{n} \Phi_i^h(v_i - z_i) \quad \text{for} \quad \boldsymbol{v} \in U,\ z_i = 0,\ i = 0, \ldots, n.$$

Here $J_0(\boldsymbol{v}) = \hat{a}^h(\boldsymbol{v}, \boldsymbol{v})/2 - \hat{l}^h(\boldsymbol{v})$ and $\boldsymbol{z}$ is the vector of the auxiliary variables z_i.

Application of the modified-Lagrangian method under the constraints $z_i = 0$ reduces to minimization of the functional

$$L_r(\boldsymbol{v}, \boldsymbol{z}, \lambda) = J_0(\boldsymbol{v}) + \sum_{i=0}^{n} \Big(\Phi_i^h(v_i - z_i) + \lambda_i, z_i + \frac{1}{2}\, r z_i^2 \Big) \quad \text{for} \quad \boldsymbol{v} \in U.$$

Minimizing the functional $L_r(\boldsymbol{v}, \boldsymbol{z}, \lambda)$ explicitly with respect to z and using the equivalence property [16]

$$\inf_{z_i} \Big(\Phi_i^h(v_i - z_i) + (\lambda_i, z_i)_{\mathbb{R}^k} + \frac{1}{2}\, r |z_i|^2_{\mathbb{R}^k} \Big) = \sup_{z_i^*} \Big((v_i, z_i^*)_{\mathbb{R}^k} - \Phi_i^*(z_i^*) - \frac{1}{2r}\, |z_i^* - \lambda_i|^2_{\mathbb{R}^k} \Big),$$

where $\Phi_i^*(\,\cdot\,)$ is a function conjugate to $\Phi_i^h(\,\cdot\,)$ such that

$$\Phi_i^*(z_i^*) = \begin{cases} 0, & z_i^* \in \bar{K}_i, \\ \infty, & z_i^* \notin \bar{K}_i, \end{cases}$$

we obtain the modified Lagrangian

$$L_r(\boldsymbol{v}, \lambda) = J_0(\boldsymbol{v}) + \sum_{i=0}^{n} \Big(\frac{1}{2}\, r v_i^2 + v_i \lambda_i - \frac{1}{2r}\, d^2(\lambda_i + r v_i, \bar{K}_i) \Big).$$

Here $d(x, A)$ is the distance from the point x to the set (segment) A on the number line $\mathbb{R}$.

Minimization of the nonsmooth functional reduces to the successive solution of problems of minimization of the modified Lagrangian $L_r(\boldsymbol{v}, \lambda)$ with respect to the vector of variables $\boldsymbol{v}$ for fixed values of the vector of multipliers λ.

To recalculate the multipliers, one can use the duality theory. According to this theory, determination of the vector of Lagrange multipliers that ensures the optimum of the problem of nonlinear programming is equivalent to maximization of the dual functional

$$D_r(\lambda) = \min_{\boldsymbol{v} \in U} L_r(\boldsymbol{v}, \lambda) = L_r(\boldsymbol{v}(\lambda, r), \lambda),$$

where $\boldsymbol{v}(\lambda, r)$ is the solution that minimizes the modified Lagrangian for specified λ.

The derivatives of the modified Lagrangian with respect to $\boldsymbol{v}$ and λ have the form

$$\nabla_v L_r(\boldsymbol{v}, \lambda) = \nabla J_0(\boldsymbol{v}) + \boldsymbol{P}(\boldsymbol{v}, \lambda, r), \qquad \nabla_\lambda L_r(\boldsymbol{v}, \lambda) = -(\lambda - \boldsymbol{P}(\boldsymbol{v}, \lambda, r))/r.$$

Here $\boldsymbol{P}(\boldsymbol{v}, \lambda, r)$ is the column vector

$$\boldsymbol{P}(\boldsymbol{v}, \lambda, r) = \left\{ \begin{array}{c} P_{\bar{K}_0}(\lambda_0 + r v_0) \\ \vdots \\ P_{\bar{K}_n}(\lambda_n + r v_n) \end{array} \right\}$$

and $P_K(z)$ is the projection of the point z onto the set K.

Taking into account the relation $\nabla_v L_r(\boldsymbol{v}, \lambda)\Big|_{\boldsymbol{v} = \boldsymbol{v}(\lambda, r)} = 0$, we express $\nabla D_r(\lambda)$ in the form

$$\nabla D_r(\lambda) = \nabla_\lambda L_r(\boldsymbol{v}, \lambda)\Big|_{\boldsymbol{v} = \boldsymbol{v}(\lambda, r)}.$$

Using the steepest ascend method to maximize the dual function, we obtain the formula for recalculation of the Hestenes–Powell multipliers

$$\lambda^{(k+1)} = \lambda^{(k)} + r \nabla D_r(\lambda^{(k)}),$$

which takes the following form for the problem considered:

$$\lambda^{(k+1)} = \boldsymbol{P}(\boldsymbol{v}(\lambda^{(k)}, r), \lambda^{(k)}, r).$$

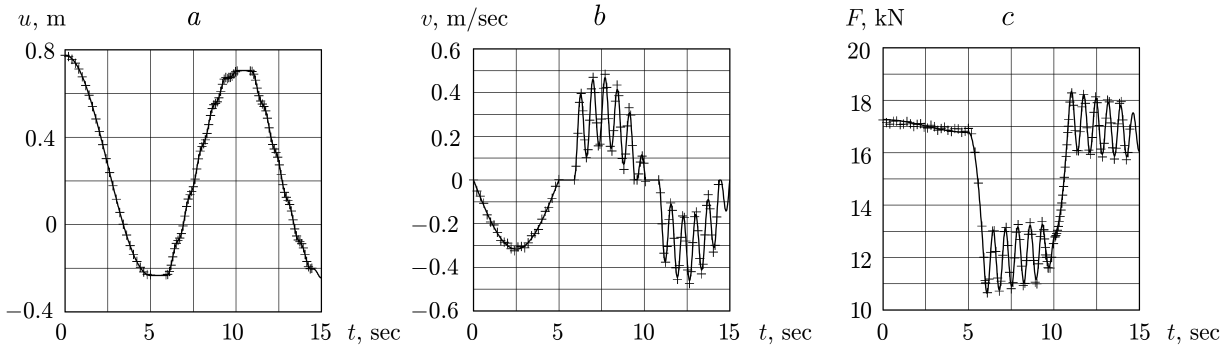

Fig. 2. Time dependences of the displacement (a), the velocity of the localized mass (b), and the force at the suspension point (c): solid curves and points refer to the numerical and analytical solutions, respectively.

Minimization of $L_r(\boldsymbol{v}, \lambda)$ with respect to $\boldsymbol{v}$ for given λ is performed by solving a system of equations that represents the minimum condition. Since constraint (11) is simple, this condition takes the form

$$\nabla_{v,FR} L_r(\boldsymbol{v}, \lambda) = 0, \qquad v_0 = \hat{u},$$

where $\nabla_{v,FR}$ is the gradient with respect to the free variables v_i. For the finite-element formulation, this relation becomes

$$K_{*,M}\boldsymbol{v} - \boldsymbol{b}_{*,M} + \boldsymbol{P}_M(\boldsymbol{v}, \lambda, r) = 0. \tag{12}$$

Here $K_{*,M}$ and $\boldsymbol{b}_{*,M}$ are the reduced stiffness matrix K_* and force vector $\boldsymbol{b}_*$, respectively, modified to take into account kinematic constraints. The vector $\boldsymbol{P}_M(\boldsymbol{v}, \lambda, r)$ is obtained by setting to zero the component of the vector $\boldsymbol{P}(\boldsymbol{v}, \lambda, r)$ that corresponds to a specified displacement.

To solve system (12), we use the Picard method. For the problem considered, the internal iterations of this method are written as

$$\boldsymbol{v}^{(k+1)} = K_{*,M}^{-1}(\boldsymbol{b}_{*,M} - \boldsymbol{P}_M(\boldsymbol{v}^{(k)}, \lambda, r)).$$

Minimization of $L_r(\boldsymbol{v}, \lambda)$ with respect to $\boldsymbol{v}$ for given λ can be performed approximately. If one iteration of the Picard method is performed for approximate minimization and the Hestenes–Powell formula is used to recalculate the multipliers, the solution procedure reduces to the known Uzawa method [9].

The penalty parameter r is found from the condition of convergence of the Uzawa iterations [9]

$$0 < r < \lambda_1(K_{*,M}),$$

where $\lambda_1(K_{*,M})$ is the minimum eigenvalue of the matrix $K_{*,M}$. If this condition is satisfied, the Picard iterations are compressive mappings.

3. Testing of the Algorithm and Its Application to Calculation of Well-Pump Rods. We verify the algorithm by comparing the numerical solution obtained by one-element discretization of the rod with the analytical solution of the system consisting of a spring and suspended mass. The suspension point performs periodic motions (in our case, according to the cosine law). The external forces $\bar{P}^-$ and $\bar{P}^+$ act on the mass as it moves up and down, respectively. The parameters of the system and the loads are chosen so that the system is equivalent to the one-element model. The algorithm was verified for the following loading and structural parameters: rod length $l = 1000$ m, rod diameter $d = 0.02$ m, period of motion of the upper end of the rod $T = 10$ sec, amplitude $U_a = 0.5$ m, $P^- = 5000$ N, and $P^+ = 0$.

The numerical solution was obtained with a time step $\Delta t = 0.025$ sec and Newmark coefficients $\beta = 0.276$ and $\gamma = 0.55$.

Figure 2 shows the displacement u, the velocity v of the localized mass, and the force F at the suspension point as functions of time. One can see from Fig. 2 that the analytical and numerical solutions almost coincide, the error in determining the displacements and forces being smaller than 1% and the error in determining the velocities being smaller than 3%.

The algorithm was used to calculate the rods of well pumps under conditions close to operation conditions. For this system, the following multivalued relations are taken into account: 1) relation between the force applied

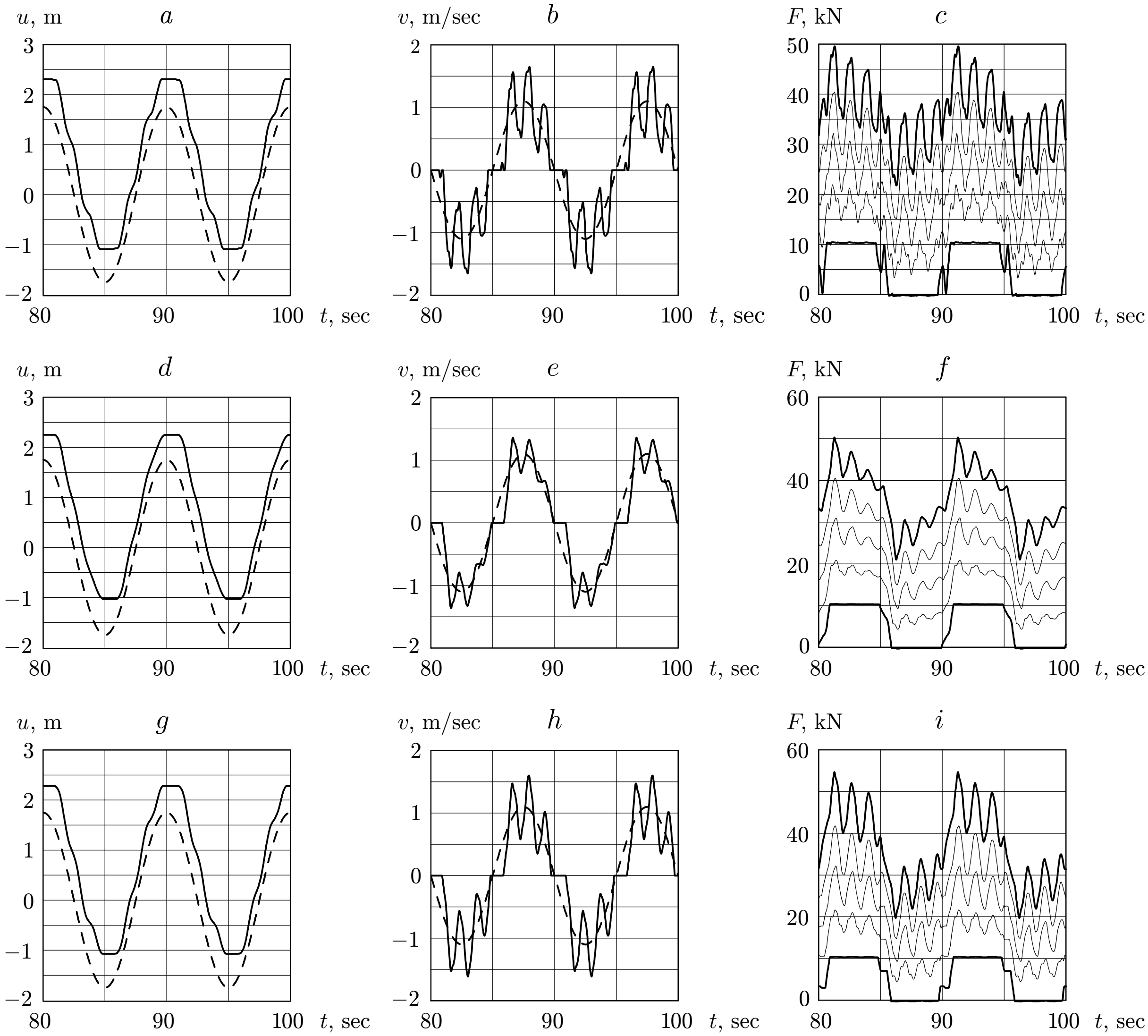

Fig. 3. Effect of resistance on the behavior of the rod: (a, b, c) no resistance; (d, e, f) hydrodynamic resistance; (g, h, i) Coulomb friction; the solid and dashed curves refer to the displacements and velocities of the lower and upper ends of the rod, respectively.

to the lower end of the rod and the velocity of this end, which depends on operation of the well-pump valves; 2) Coulomb friction of the rods on the walls of the tubing string.

The effect of the Coulomb-friction and viscous forces on the dynamic behavior of the rod was studied for the following loading conditions and structural parameters: rod length $l = 1700$ m, rod diameter $d_s = 19$ mm, tube diameter $d_t = 62$ mm, perturbation period $T = 10$ sec, amplitude of motion of the upper end of the rod $a = 3.5$ m, $P^- = 10{,}000$ N, and $P^+ = -400$ N. The upper end of the rod moves according to the cosine law.

The zenith angle describing the channel geometry varies linearly from 0 at the top of the channel to 20° at a depth of 425 m. The angle remains constant to a depth of 1275 m and then decreases linearly to 0 at a depth of 1700 m.

The density and static viscosity of the fluid pumped out are assumed to be constant: $\rho_f = 900$ kg/m^3 and $\nu = 10^{-4}$ m^2/sec, respectively. The friction coefficient of the rod on the tube walls is $f = 0.3$, Young's modulus is $E = 2 \cdot 10^5$ MPa, and the density of the rod material is $\rho = 7800$ kg/m^3.

The rod was modeled by 80 elements in the lengthwise direction, and the time step was $\Delta t = T/400$.

The solid curves in Fig. 3 show the displacements and velocities of the lower end of the rod and the axial forces at the mid-points of the 1st, 20th, 40th, 60th, and 80th elements, which correspond to the cross sections of the rod located at distances from its upper end approximately equal to 0, $L/4$, $L/2$, $3L/4$, and L, respectively. The dashed curves refer to the displacements and velocities of the upper end.

The lower end of the rod moves discontinuously; during the stops, the force acting on the lower end increases from minimum to maximum. In addition to forced oscillations, natural oscillations are also observed. The latter are excited twice per cycle as the lower end changes the direction of its motion.

The hydrodynamic resistance and Coulomb friction affect the behavior of the structure in different ways. In both cases, the amplitude of the displacement of the lower end decreases and the amplitude of the axial force in the upper cross section increases. The action of viscous forces from the fluid pumped out leads to rapid damping of free oscillations. Moreover, the amplitude of the velocity of the lower end is much lower as compared to the case where hydrodinamic resistance is ignored. The action Coulomb-friction forces only does not lead to damping of free oscillations and decreases the amplitude of velocity only slightly.

In the absence of Coulomb friction, the behavior of the curves in Fig. 3 is almost identical in the intervals of ascent and descent. In the presence of the Coulomb-friction force, this effect is not observed since these forces depend on the pressing forces, which, in turn, depend on the current state of the system.

Conclusions. Numerical verification of the algorithm proposed for solving dynamic problems with multi-valued nonlinearities and a comparison with the known analytical solution show that the algorithm yields adequate results.

The algorithm and its numerical implementation allow one to calculate the axial oscillations of deep-well pump rods and study the effect of viscous resistance and Coulomb friction on the dynamic behavior of the rods.

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